2-Categorical Poincaré Representations and State Sum Applications

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Abstract

This is intended as a self-contained introduction to the representation theory developed in order to create a Poincaré 2-category state sum model for Quantum Gravity in 4 dimensions. We review the structure of a new representation 2-category appropriate to Lie 2-group symmetries and discuss its application to the problem of finding a state sum model for Quantum Gravity. There is a remarkable richness in its details, reflecting some desirable characteristics of physical 4-dimensionality. We begin with a review of the method of orbits in Geometric Quantization, as an aid to the intuition that the geometric picture unfolded here may be seen as a categorification of this process.

1 Introduction

There has been much investigation into constrained topological state sums for Quantum Gravity in four dimensions. The motivation for these constructions follows largely from the success of 3-dimensional models, such as the Turaev-Viro state sum. The elegant category theoretic formulation of these 3-dimensional TFTs has led to a search for even richer 4-dimensional analogues.

To date, perhaps the most promising candidates for a finite theory of quantum gravity are the Lorentzian Barrett-Crane model [1] and its q-deformed version [2]. In [3] Crane and Yetter propose a new state sum for Lorentzian Quantum Gravity which utilises a higher algebraic symmetry: the Poincaré group as a 2-category on one object. In order to attempt to write down the state sum it is first necessary to more concretely understand the representation category defined in [4].

The physical motivation for our construction is the idea that the fundamental symmetry to use to construct quantum gravity is the Poincaré group action, but with the translation subgroup differentiated from the Lorentz group. Translations could correspond to lengths, and infinitesimal rotations to bivectors, the

representation category providing a quantization of the whole system of geometrical quantities. More mathematically expressed, the Poincaré group admits a canonical decomposition

$$\mathbf{M}^4 \to \mathbf{P} \to \mathbf{L}$$

where $\mathbf{M}^4$ is the additive group of translations in $\mathbb{R}^4$ and $\mathbf{L}$ is the (connected) Lorentz group $SO(3,1)$; and we need a new notion of representation theory which respects this decomposition. This information is lost in treating the Poincaré group simply as a group.

The way we accomplish this is to regard the Poincaré group as naturally a strict 2-group, which we call **Poinc**, as follows. 1-morphisms are elements of $\mathbf{L}$. 2-morphisms $g_1 \to g_2$ are elements $x \in \mathbf{M}^4$ that map to $g_1^{-1}g_2$, which in this case takes across all of $\mathbf{M}^4$ when $g_1 = g_2$ and nothing otherwise, as in the globule

$$\begin{tikzcd} * \arrow[r, bend left, "g"{name=U}] \arrow[r, bend right, "g"'{name=D}] & * \arrow[Rightarrow, from=U, to=D, "x"] \end{tikzcd}$$

The action $\alpha : \mathbf{L} \times \mathbf{M}^4 \to \mathbf{M}^4$ appears in the 2-category as part of the tensor structure, as does the target map t taking $\mathbf{M}^4$ trivially to the identity in $\mathbf{L}$.

Higher category theory [5][6] has long been discussed in the context of 4-dimensional TQFTs. In a more immediately physical setting it will be interesting to investigate, for instance, Higher Yang-Mills theories [7], or Higher Lattice Gauge theory [8]. Unfortunately, the apparent complexity in the detailed definitions of higher categories is liable to lead to the misconception that they are ad hoc and inelegant abstractions. In fact, the construction outlined here is quite canonical in that it all falls out of one coherent algebraic structure. We hope to make clear that the new representation 2-category is actually a natural object in differential geometry.

The substitution of a 2-group for an ordinary group is an example of the process of categorification [9], which has been believed to play an important role in raising the dimension of categorically constructed theories for some time. Thus the suggestion that we use a 2-group as the fundamental symmetry of a 4D theory is very natural.

In order to understand our new construction, it is useful to first adopt a slightly novel point of view on the representation theory of ordinary groups. Any group may be thought of as a category, with one object $*$, with group elements described as invertible morphisms (arrows). Let us refer to groups so thought of as 1-groups. What does the representation theory for 1-groups involve? The usual idea of describing a representation by an action of the group on some vector space is replaced by the natural categorical analog: a functor from the 1-group category into the tensor category of vector spaces. A morphism between such functors F and G is a *natural transformation* [10]: for all arrows g there exists n such that

$$\begin{tikzcd} F(*) \arrow[r, "F(g)"] \arrow[d, "n"'] & F(*) \arrow[d, "n"] \\ G(*) \arrow[r, "G(g)"'] & G(*) \end{tikzcd}$$

commutes.

The reader should verify that this corresponds to the usual ideas of group representations and intertwining operators.

Now one would like to construct an appropriate 2-categorical analog of the above for **Poinc**. The most natural suggestion would be to consider the category of functors between **Poinc** and a suitable 2-categorical analog of **Vect**. Previously studied representation 2-categories, such as **2-Vect** [6] or **2-Hilb** [11], suffer from the fact that the corresponding Poincaré 2-group representation theory admits very few representations (as the Lorentz group is not profinite).

In order to remedy this problem, a new 2-category called **Meas** has been constructed, which we describe below. The main purpose of this paper is to spell out the concrete structure of the new 2-category of representations of the Poincaré 2-group in **Meas**, introduced in [4], [12] and [13]. As we shall see, it is very rich. A large part of the subjects of harmonic analysis, group representation theory, mathematical physics and low dimensional topology are combined into a unified algebraic structure in a peculiarly categorical way. In particular, we observe the existence of a smooth subcategory closed under tensor product.

Whereas a 1-category has objects and morphisms, a 2-category has objects, 1-morphisms and 2-morphisms. For objects A and B of a 2-category, $\mathrm{Hom}(A, B)$ forms a category. For example, **Toph** is the 2-category of topological spaces, homeomorphisms and homotopies.

The category of functors between two 2-categories is itself a 2-category, with functors, weak or strict natural transformations, and modifications. Thus the new representation 2-category has three levels of structure which need to be made explicit.

Interesting higher categories are not strict but *weakened*: coherence relations for a category may hold only *up to* a 2-morphism, and coherence relations become higher dimensional polytopes. It is precisely the possibility of weakening morphisms that gives a much richer symmetry out of which one might hope to build interesting state sums. We shall see below that the weak natural transformations (1-intertwiners) of our 2-category are richer geometric objects than the strict ones.

Now let us briefly describe the 2-category **Meas**. The objects of this category [4][13] are the categories of measurable fields of Hilbert spaces over a given measure space. 1-intertwiners are fields of Hilbert spaces on the product of the domain and range measure spaces, together with a measure on the product. These should be thought of as matrices with continuum indices, the measure allowing one to sum up matrix elements when composing. The 2-intertwiners are measurable fields of Hilbert space operators.

A single representation is now a 2-functor from the 2-group into the range 2-category **Meas**, and the higher category of all representations is built out of these functors with (weakened) pseudo-natural transformations $\{n(X) : F \to G\}$ and modifications $\mu : n \Rightarrow m$ as morphisms. The basic coherence relations

for these morphisms are

$$
\begin{tikzcd}
F(X) \arrow[r, bend left, "F(f)"] \arrow[r, bend right, "F(g)"'] \arrow[r, phantom, "\Downarrow"] \arrow[d, "n(X)"'] \arrow[dr, phantom, "\Downarrow n(g)"] & F(Y) \arrow[d, "n(Y)"] \\
G(X) \arrow[r, "G(g)"'] & G(Y)
\end{tikzcd}
\quad \simeq \quad
\begin{tikzcd}
F(X) \arrow[r, "F(f)"] \arrow[d, "n(X)"'] \arrow[dr, phantom, "\Downarrow n(f)"] & F(Y) \arrow[d, "n(Y)"] \\
G(X) \arrow[r, bend left, "G(f)"] \arrow[r, bend right, "G(g)"'] \arrow[r, phantom, "\Downarrow"] & G(Y)
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
F(X) \arrow[r, bend left, "F(f)"] \arrow[r, bend right, "F(g)"'] \arrow[r, phantom, "\Downarrow"] \arrow[d, bend right, "m(X)"'] \arrow[d, bend left, "n(X)"] \arrow[d, phantom, "\overset{\Leftarrow}{\mu(X)}"] \arrow[dr, phantom, "\Downarrow n(g)"] & F(Y) \arrow[d, "n(Y)"] \\
G(X) \arrow[r, "G(g)"'] & G(Y)
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
F(X) \arrow[r, "F(f)"] \arrow[d, "m(X)"] \arrow[dr, phantom, "\Downarrow m(f)"] & F(Y) \arrow[d, bend right, "m(Y)"'] \arrow[d, bend left, "n(Y)"] \arrow[d, phantom, "\overset{\Leftarrow}{\mu(Y)}"] \\
G(X) \arrow[r, bend left, "G(f)"] \arrow[r, bend right, "G(g)"'] \arrow[r, phantom, "\Downarrow"] & G(Y)
\end{tikzcd}
$$

By taking the direct integral of the Hilbert spaces in a 1-intertwiner of **Meas**, a representation of the ordinary Poincaré group is obtained. However, irreducibles do not go to irreducibles, so the structure of the representation 2-category is new. Despite some technical resemblance it does not by any means reduce to the famous Wigner classification.

We begin by reviewing Kirillov's *method of orbits* for 1-groups, as it clarifies somewhat the structure of our new representation category, which may be thought of as a categorification of this process. The analogy between the structure of the new category and the method of orbits is quite striking. In place of coadjoint orbits one finds two levels of orbits: orbits in Minkowski space and fibrations of Lorentz orbits over these. The tensor product in our new category is a geometric operation in two steps: the lower level is very close to the operation of Kirillov. Higher dimensional versions of Geometric Quantization should make explicit what one means by the *quantum geometry* of 2-categorical state sums [14].

A note on notation: Objects and functors are denoted by capital Latin letters, 1-morphisms by lower-case $f : A \to B$, and 2-morphisms by, for example, $\mu : f \Rightarrow g$. Monoidal categories have identities I with respect to the product. The identity arrow of an object A is 1_A.

2 Geometric Quantization

The simplest physical example of Geometric Quantization is the quantization of phase space $\simeq \mathbb{R}^{2n}$ in field theory. This manifold may be endowed with a symplectic structure, and all symplectic manifolds look locally like a patch of

this space with coordinates $(x_1, x_2, \cdots, x_n, p_1, p_2, \cdots, p_n)$. In this section we briefly outline the general construction.

Let $\mathcal{G}$ denote the Lie algebra of a semi-simple Lie group $\mathbf{G}$. For our purposes, $\mathbf{G}$ is $SL(2,\mathbb{C})$, which is locally isomorphic to the Lorentz group $\mathbf{L}$. An *orbit* of the $\mathbf{G}$-action on a space M has the form of a symmetric space

$$\mathcal{O} = G/G_x$$

for G_x the isotropy at a point $x \in M$.

The coadjoint orbits of $\mathbf{G}$ [15][16] have a natural symplectic structure under the action of $\mathbf{G}$, and those orbits satisfying an integrality condition induce unitary irreducible representations of $\mathbf{G}$. In fact, the unitary irreps appearing correspond to the decomposition of the regular representation.

Functions on $\mathcal{G}^*$, corresponding to elements of the Lie algebra, satisfy a Kirillov-Poisson bracket dual to the Lie algebra structure [15]. For a basis X_i of $\mathcal{G}$ and structure constants c_{ij}^k this bracket is given by

$$[f_1, f_2] = \sum_{i,j,k} c_{ij}^k X_k \frac{\partial f_1}{\partial X_i} \frac{\partial f_2}{\partial X_j} \tag{2.1}$$

For vector fields on $\mathcal{G}^*$

$$v_i = \sum_{j,k} c_{ij}^k X_k \frac{\partial}{\partial X_j} \tag{2.2}$$

there is a canonical symplectic structure on each coadjoint orbit given by

$$\omega(v_i, v_j) = \sum_k c_{ij}^k X_k \tag{2.3}$$

Generically, full quantization is possible if there exists a connection ∇ on a line bundle over the orbit [16] such that the curvature satisfies

$$F(\nabla) = 2\pi i \omega \tag{2.4}$$

In other words, the orbit is *integral*. This means that the 1-dimensional representation of a certain Lie algebra $\mathcal{H} \subset \mathcal{G}$ may be extended to a unitary representation of the corresponding group $\mathbf{H}$, and it is from this representation that a unitary irrep of $\mathbf{G}$ is induced. This induction step is functorial. That is, we are hoping to view the representation 2-category in its geometric guise entirely within the categorical formalism.

The integrality corresponds to the fact that $\omega \in H^2(M,\mathbb{Z})$, suggesting perhaps that an appropriate 2-categorical analogue of quantization would involve higher cohomological conditions, such as in the theory of gerbes. This question will not be addressed here.

Explicitly, for the case of $\mathcal{G} = sl(2,\mathbb{C})$ the orbits on $\mathcal{G}^*$ are derived as follows. Let a, b, c, d denote complex variables such that $ad - bc = 1$. Under the identification $\mathcal{G} \sim \mathcal{G}^*$ the coadjoint orbits of $SL(2,\mathbb{C})$ are given by the action of

$$\begin{pmatrix} d^2 & cd & c^2 \\ 2bd & ad+bc & 2ac \\ b^2 & ab & a^2 \end{pmatrix}$$

on $\mathbb{C}^3$. After a suitable change of coordinates, in $\mathbb{R}^6$ these orbits are given by the zero orbit and intersections

$$x_0^2 + x_1^2 + x_2^2 - y_0^2 - y_1^2 - y_2^2 = n^2 - \rho^2 \tag{2.5}$$

$$x_0 y_0 + x_1 y_1 + x_2 y_2 = n\rho \tag{2.6}$$

for $n, \rho \in \mathbb{R}$. Clearly these curves foliate $\mathbb{R}^6$. Thus we have a remarkably easy classification of the equivalence classes of unitary representations $\pi_{n\rho}$ for $SL(2,\mathbb{C})$ [17].

Integrality demands that $n \in \frac{1}{2}\mathbb{Z}$, as for $SU(2)$, which has spherical orbits in $su(2)^* \simeq \mathbb{R}^3$. (In the simpler case of $SU(2)$, the integral orbits correspond to the well known integral levels of total quantized angular momentum, making rigorous the physicist's intuition about adding quantum angular momenta by adding vectors with uncertain direction).

The irreps appearing in the decomposition of a tensor product of irreps can be recovered geometrically from the sum of orbits

$$\mathcal{O}_1 + \mathcal{O}_2 \equiv \{x_1 + x_2 : x_1 \in \mathcal{O}_1, x_2 \in \mathcal{O}_2\}$$

In the case of $SU(2)$, the decomposition $\mathcal{H}_j \otimes \mathcal{H}_l = \bigoplus_{|j-l|}^{|j+l|} \mathcal{H}_i$ into spherical shells follows from the range of the norm of the sum of two vectors of lengths j and l. For $SL(2,\mathbb{C})$ we obtain

$$\pi_{n_1\rho_1} \otimes \pi_{n_2\rho_2} = \bigoplus_{m+n_1+n_2 \in \mathbb{Z}} \int^{\oplus} \pi_{m\rho} d\rho \tag{2.7}$$

This so-called *method of orbits* [15] is seen to describe the structure of the tensor category of representations of a Lie group, as a geometric category whose tensor product is a geometric operation.

The usual unitary representations of the Poincaré group can be similarly described by the orbits in Minkowski space along with a representation of the stabiliser glued to each point of the orbit.

3 2-Representations

We now outline the structure of the representation 2-category, noting its form as a two stage Geometric Quantization. Its elements are

- **Objects:** 2-functors, labelled by a module object and an action upon it by $\mathbf{L}$ and $\mathbf{M}^4$.
- **1-morphisms:** pseudo-natural transformations
- **2-morphisms:** modifications

The general construction, for any 2-group, was introduced in [4] and [13]. Let $\mathbf{Meas}(X)$ denote the 1-category whose objects are measurable fields of Hilbert spaces $\{\mathcal{H}_x\}$ indexed by the Borel space X. The 1-morphisms are all bounded fields of bounded operators between fields of Hilbert spaces. This is well defined if we say a field of bounded operators is bounded when $x \mapsto \|\phi_x\|$, for $\phi_x \in \mathcal{B}(\mathcal{H}_x, \mathcal{K}_x)$, is a bounded real function. The spaces X may be thought

of as continuous generalizations of the discretely labelled categories **Vect-n** underlying **2-Vect**.

A single representation is a 2-functor $\mathcal{R} : \mathbf{Poinc} \to \mathbf{Meas}$ from the 2-group to the 2-category of all such categories $\mathbf{Meas}(X)$. The unique object is mapped to some specific measure space X, and the morphisms, thought of as a category, get mapped to objects and morphisms in the hom category $\mathbf{Meas}(X, X)$.

More concretely, each element g of $\mathbf{L}$ is sent to a field of Hilbert spaces on $X \times X$, which must obey the group law under convolution, and in particular be invertible. In order for a field of Hilbert spaces to be invertible, each Hilbert space must be one dimensional, and there can only be one non-zero Hilbert space in each horizontal or vertical line in $X \times X$. Thus, the representation of the 1-intertwiners of **Poinc** means that we are given a measurable action of $\mathbf{L}$ on the measure space X.

Now let us consider the images of the 2-intertwiners of **Poinc**. Each vector of $\mathbf{M}^4$ is assigned a linear map on each Hilbert space in the field corresponding to each $g \in \mathbf{L}$. These must satisfy the (additive) group law of the vector space $\mathbf{M}^4$. This means that each point in the graph of the action of each $g \in \mathbf{L}$ is assigned a character, which may be identified with a point in $\mathbf{M}^4$. The group laws of **Poinc** now imply that the characters are determined by the characters on the graph of the identity functor on X, and that they are equivariant with respect to the action of $\mathbf{L}$ on X.

This translates into the following:

Proposition 3.1 *Objects of the 2-category of representations of* **Poinc** *in* **Meas** *correspond to measure spaces on which* $\mathbf{L}$ *acts measurably, provided with equivariant maps to* $\mathbf{M}^4$

For example, let X be an hyperboloid orbit of the Lorentz group in Minkowski space, such as $\mathcal{O}_\rho \equiv \{t^2 - x_1^2 - x_2^2 - x_3^2 = \rho\}$. The Lorentz group acts on $\mathcal{H}_x$ by translation on $x \in \mathcal{O}_\rho$, and the characters $\chi \in \widehat{\mathbf{M}^4}$ act as scalar multipliers on the coordinates of the point.

Now let us classify the irreducible representations. Any orbit of $\mathbf{L}$ in a representation space X is a subrepresentation. Orbits of a group in an action correspond to quotients of the group by the stabilizer subgroup of a point. In order for the orbit to admit an equivariant map to $\mathbf{M}^4$, the stabilizer subgroup must be contained in the stabilizer of some point in $\mathbf{M}^4$. The image of the quotient under the equivariant map is then some orbit of $\mathbf{L}$ in $\mathbf{M}^4$.

A simple set of elementary irreducible representations are given by the orbits $\mathcal{O} \hookrightarrow \mathbf{M}^4$, with the trivial representation attached, on which $\mathbf{L}$ acts transitively. But these are not the only irreducibles. One must include multiple copies of the same orbit of the form $M \to^\pi \mathcal{O}$, where the fibre is a symmetric space of the stabilizer for the orbit. These copies are permuted under the action of $\mathbf{L}$.

In other words, irreps correspond to orbits M of $\mathbf{L}$-actions α which are equivariant fiberings

$$\begin{tikzcd} M \arrow[r, "\alpha"] \arrow[d, "\pi"'] & M \arrow[d, "\pi"'] \\ \mathcal{O}_\rho \arrow[r] & \mathcal{O}_\rho \end{tikzcd}$$

Because these representations are in 1 : 1 correspondence with subgroups $H \subseteq G_\rho$ for each Minkowski orbit type, we obtain a rather large class of irreducibles, which contains

1. **Elementary Irreps:** correspond to the case where the stabilizer of M equals the stabilizer of the orbit $\mathcal{O}_\rho$ of $\mathbf{M}^4$ over which it fibers. Denoted by E_ρ.

2. **Lie Irreps:** occur where the subgroup of the stabilizer of the orbit $\mathcal{O}_\rho$ is a connected Lie group. For example, if $\mathcal{O}_\rho$ is a spacelike hyperboloid, its stabilizer is $SU(2)$. One could either take S^1 for the subgroup, in which case M is an S^2 bundle over the hyperboloid, or let the stabilizer be trivial, in which case the space M is a copy of the group $\mathbf{L}$, which can be written as an S^3 bundle over the hyperboloid. This maximal irrep, which can occur over any orbit in $\mathbf{M}^4$, we denote by L_ρ.

3. **Crystallographic Irreps:** occur if the stabilizer of M contains a discrete subgroup of the stabilizer of the orbit in $\mathbf{M}^4$. We obtain an irrep whose fiber over a point in $\mathbf{M}^4$ is a manifold whose fundamental group includes the given discrete group.

4. **Non-Hausdorff Irreps:** occur when we choose as stabilizer a non-Lie subgroup of $\mathbf{L}$ to produce a total space which is not Hausdorff.

There is a subset of irreps, namely the first three cases, which inherit a natural smooth structure. This subcategory will close under tensor product and direct integral by a smooth index space. The study of the non-Hausdorff irreps will require very different mathematical tools.

Proposition 3.2 *For all orbits of* $\mathbf{L}$ *in* $\mathbf{M}^4$*, the 4 cases above exhaust the irreducible objects of* $\mathbf{Rep}(\mathbf{Poinc})$

3.1 Tensor Products of Objects

The tensor product of two objects M_1 and M_2 in **Meas** corresponds to the cartesian product of the underlying measure spaces. The 2-group has a natural action on the tensor product of two representations, where the functors corresponding to elements of $\mathbf{L}$ act in both variables at once, while the actions of the 2-morphisms of **Poinc** on the spaces over a point in $M_1 \times M_2$ is the tensor product of the respective actions. Note that the elementary zero orbit E_0 acts as an identity element.

Now the tensor product of two characters of the abelian group $\mathbf{M}^4$ just corresponds to vector addition. Thus the tensor product of two objects of $\mathbf{Rep}(\mathbf{Poinc})$ corresponds to taking the sum of their two projections as subsets of the vector space $\mathbf{M}^4$ and projecting the product space $M_1 \times M_2$ into $\mathbf{M}^4$ by sending each point to the vector sum of the projections of its two coordinates. Note the analogy with the method of orbits.

The decomposition of tensor products into direct integrals of irreps is easy enough to work out explicitly in the simpler cases, uncovering an interesting picture. Whereas products of basis elements in an algebra are determined by

structure coefficients, and tensor products in a category based in **2-Vect** are defined by structure vector spaces, the tensor product of irreps in **Rep**(**Poinc**) decomposes in terms of structure spaces. These structure spaces are measure spaces in general, but smooth manifolds for the most natural cases. Observe that in this 2-categorical theory, representations are therefore *not* linear at all levels. Linearity is only required at the top level, where we wish to define trace operators.

We denote this type of direct integral

$$\int^{\uplus} (M, F) \tag{3.1}$$

where F is the fibre of M.

Let us illustrate this by means of a simple but important example: the case of the tensor product of two elementary irreps, corresponding to orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ in $\mathbf{M}^4$. The fibers over each point of each orbit are single points. Therefore, the fiber of the tensor product over any point of $\mathbf{M}^4$ is the set of all ways to decompose the point as a sum of two points, one in $\mathcal{O}_1$ and one in $\mathcal{O}_2$. The decomposition into irreps is accomplished by first decomposing into orbits in $\mathbf{M}^4$, then decomposing the fiber over a generic point into orbits of the stabilizer of the orbit.

Consider the case where both $\mathcal{O}_1$ and $\mathcal{O}_2$ are spacelike hyperboloids with radii ρ_1 and ρ_2. The set-theoretic sum $\mathcal{O}_1 + \mathcal{O}_2$ is the union of all spacelike hyperboloids with radius $\rho \geq \rho_1 + \rho_2$. The fiber over each point is the set of all timelike triangles in $\mathbf{M}^4$ with side lengths (ρ_1, ρ_2, ρ). Generically this set forms a 2-sphere, which is a single orbit of $SU(2)$. Thus the tensor product of two spacelike irreps is a direct integral of Lie irreps with stabilizer S^1

$$E_{\rho_1} \otimes E_{\rho_2} = \left(\int^{\uplus}_{\rho > \rho_1 + \rho_2} (M_\rho, S^2)\right) + E_{\rho_1 + \rho_2} \tag{3.2}$$

The second term corresponds to the collinear (ρ_1, ρ_2, ρ) case.

Extending this procedure to the tensor product of several elementary irreps allows us to compute the tensor product of the various Lie irreps.

The triple tensor product contains copies of the maximal irreps L_ρ of allowable orbits because it contains the space of quadrilaterals with appropriate edge lengths (figure 1). In other words, a generic quadrilateral in $\mathbf{M}^4$ admits no isotropy subgroup. In this decomposition there is not simply one maximal irrep, but rather a family, indexed by the space of shapes of such quadrilaterals in $\mathbf{M}^4$. Copies of (M_ρ, S^2) arise for planar quadrilaterals, of isotropy group $SO(2)$.

Observe that the structure spaces for such cases are smooth, and the irreps that appear in the decomposition of tensor products of fundamental irreps are also smooth. This suggests that it is possible to restrict to a smooth subcategory, which might be expedient in applications. Moreover, in studying deformations, or cohomology, of our category, in the smooth case everything will reduce to a tractable study of suitable forms on products of shape spaces.

The tensor products of all Lie irreps will be worked out in detail in later papers. The problem for crystallographic irreps will need slightly different methods. In some cases, this family is quite large, i.e. corresponds to Fuchsian groups. It is remarkable that an algebraic structure combining them is defined canonically. The non-Hausdorff irreps will probably be harder to describe explicitly. They do not appear in the tensor products of the smooth representations.

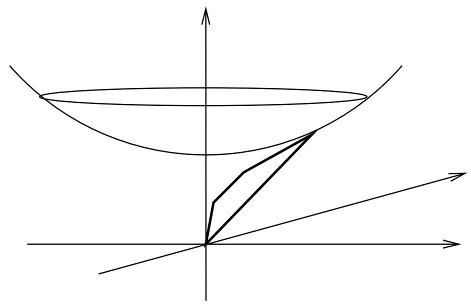

Figure 1: quadrilateral in $\mathbf{M}^4$

4 1-Intertwiners

A *strong* 1-intertwiner between two objects in $\mathbf{Rep}(\mathbf{Poinc})$ is given by a field of Hilbert spaces on the product of their underlying measure spaces, which is invariant under the product group operation and confined to ordered pairs which fiber over the same point x in $\mathbf{M}^4$. That is,

$$\mathcal{H}_x \mapsto \int dx \;\; \mathcal{H}_x \otimes \mathcal{K}_{(x,y)} d\mu_y(x) \tag{4.1}$$

which is a translation of

$$\begin{tikzcd}
X \arrow[r, bend left, "\mathcal{R}_1(f)"] \arrow[r, bend right, "\mathcal{R}_1(g)"'] \arrow[r, phantom, "\Downarrow"] \arrow[d] \arrow[dr, phantom, "\Downarrow n(g)"] & X \arrow[d] \\
Y \arrow[r, "\mathcal{R}_2(g)"'] & Y
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "\mathcal{R}_1(f)"] \arrow[d] \arrow[dr, phantom, "\Downarrow n(f)"] & X \arrow[d] \\
Y \arrow[r, bend left, "\mathcal{R}_2(f)"] \arrow[r, bend right, "\mathcal{R}_2(g)"'] \arrow[r, phantom, "\Downarrow"] & Y
\end{tikzcd} \tag{4.2}$$

A *weak* 1-intertwiner has the same structure, together with a linear action of the isotropy group of x on the Hilbert space assigned to it.

Thus, in order to obtain an explicit description of the irreducible 1-intertwiners between two representations we decompose the cartesian product of the underlying measure spaces into group orbits and describe the orbits and the orbit space. In the weak case we include irreducible representations of the isotropy group as well. The general (weak) 1-intertwiner is then a direct integral of any measurable family of points in the orbit space.

In other words, for irreps of the form

$$\begin{tikzcd}
F_i \arrow[r] & M_i \arrow[d, "\pi"'] \\
 & \mathcal{O}_\rho
\end{tikzcd}$$

which are symmetric spaces for $\mathbf{L}$, 1-intertwiners $f : M_1 \to M_2$ must be equivariant with respect to the action of $\mathbf{L}$, i.e. f is concentrated on the fibres F_1

and F_2. Let

$$F_1 \sim \frac{G_{\mathcal{O}_\rho}}{G_1} \qquad\qquad F_2 \sim \frac{G_{\mathcal{O}_\rho}}{G_2}$$

Then intertwiners are built on bridges

$$\begin{tikzcd} & \mathcal{I} \arrow[dl] \arrow[dr] & \\ F_1 & & F_2 \end{tikzcd} \tag{4.3}$$

for

$$\mathcal{I} \sim \frac{G_{\mathcal{O}_\rho}}{G_1 \cap G_1}$$

In particular, note that the space of irreducible 1-intertwiners between two smooth irreps is smooth. This further justifies the idea of a smooth subcategory.

In the simplest case of 1-intertwiners from an elementary irrep E_ρ to itself, we obtain only the constant functions on the orbit, since coherence implies the pointwise condition

$$n(g_1, x)n(g_2, g_1(x)) = n(g_1 g_2, x) \tag{4.4}$$

and we are free to choose a basis at each $x \in E_\rho$ under which n must be invariant, forcing $n \equiv 1$.

This is very suggestive with regards to the problem of construction of state sum models: the weak 1-intertwiners are the space of integrable spinors at a point, i.e. the representations of the isotropy group at the point. For example, for $(M_\rho, S^2) \to (M_\rho, S^2)$ we obtain fields of measurable subsets of S^2.

The composition of 1-intertwiners requires that the convolution of two orbits be decomposed into orbits. For tensor product, the product of orbits must be decomposed into orbits. In general, the tensor product of two 1-intertwiners is defined weakly by a choice of 2-intertwiner [18]

$$\begin{tikzcd} X_1 \otimes Y_1 \arrow[r, "X_1 \otimes g"] \arrow[d, "f \otimes Y_1"'] \arrow[dr, phantom, "\overset{f\otimes g}{\Rightarrow}"] & X_1 \otimes Y_2 \arrow[d, "f \otimes Y_2"] \\ X_2 \otimes Y_1 \arrow[r, "X_2 \otimes g"'] & X_2 \otimes Y_2 \end{tikzcd} \tag{4.5}$$

4.1 Internal Hom Functor and Duality

The new 2-category has, modulo some subtleties about measures, an internal Hom functor and a duality. These satisfy the usual relationship with the tensor product

$$\mathrm{Hom}(A, B \otimes C) = \mathrm{Hom}(A \otimes B^*, C) \tag{4.6}$$

The duality comes from reflection about the origin in $\mathbf{M}^4$. The internal Hom functor comes from identifying equivariant Hilbert fields with Hilbert fields over the space of orbits under $\mathbf{L}$.

In the case of two irreducibles fibering over the same orbit in $\mathbf{M}^4$, the orbit space should be thought of as fibering over the origin in $\mathbf{M}^4$. This makes perfect sense in terms of the above formula, if we let C be the trivial representation E_0.

The subtlety is that a 1-intertwiner requires a choice of measure. If we only want to work in the smooth subcategory, we can restrict to the usual Lebesgue measures on the appropriate spaces and the above relation becomes rigorous.

These structures are necessary for the detailed definition of a state sum model.

5 2-Intertwiners

A 2-morphism in **Meas** is defined as a measurable field of operators on the cartesian product space on which the two 1-morphisms are defined, from one field of Hilbert spaces to the other. In order for a 2-morphism between two 1-intertwiners to be a modification of functors, and hence a 2-intertwiner in $\mathbf{Rep}(\mathbf{Poinc})$, it must intertwine the characters $\chi \in \widehat{\mathbf{M}^4}$. Thus, a 2-intertwiner between irreducible 1-intertwiners is given by a measurable scalar function on the orbit in the cartesian product, where it is defined. This all takes place within a fiber over $\mathbf{M}^4$. The horizontal and vertical compositions

$$
\begin{tikzcd}
X \arrow[r, bend left, "\mathcal{R}(f)"] \arrow[r, bend right, "\mathcal{R}(f)"'] \arrow[r, phantom, "\Downarrow"] & X \arrow[r, bend left, "\mathcal{R}(g)"] \arrow[r, bend right, "\mathcal{R}(g)"'] \arrow[r, phantom, "\Downarrow"] & X
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X \arrow[r, bend left=60, "\mathcal{R}(f)"] \arrow[r] \arrow[r, bend right=60, "\mathcal{R}(f)"'] \arrow[r, phantom, shift left=3, "\Downarrow"] \arrow[r, phantom, shift right=3, "\Downarrow"] & X
\end{tikzcd}
$$

are defined by convolutions. For example, for two constant function intertwiners on $\mathcal{O}_\rho$, convolution is the usual convolution of functions.

The conditions on 2-intertwiners from tensor products [19] are

$$
\begin{tikzcd}
X_1 \otimes Y_1 \arrow[r, bend left, "X_1\otimes f"] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow X_1\otimes\beta"] \arrow[d, bend right, "g\otimes Y_1"'] \arrow[d, bend left] \arrow[d, phantom, "\Leftarrow \alpha\otimes Y_1"] \arrow[dr, phantom, "\Downarrow h\otimes k"] & X_1 \otimes Y_2 \arrow[d] \\
X_2 \otimes Y_1 \arrow[r] & X_2 \otimes Y_2
\end{tikzcd}
=
\begin{tikzcd}
X_1 \otimes Y_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow g\otimes f"] & X_1 \otimes Y_2 \arrow[d, bend right] \arrow[d, bend left, "h\otimes Y_2"] \arrow[d, phantom, "\Leftarrow \alpha\otimes Y_2"] \\
X_2 \otimes Y_1 \arrow[r, bend left] \arrow[r, bend right, "X_2\otimes k"'] \arrow[r, phantom, "\Downarrow X_2\otimes\beta"] & X_2 \otimes Y_2
\end{tikzcd}
\tag{5.1}
$$

and

$$
\begin{tikzcd}
X_1 \otimes Y_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow f_1\otimes g_1"] & X_1 \otimes Y_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow f_1\otimes g_2"] & X_1 \otimes Y_3 \arrow[d] \\
X_2 \otimes Y_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow f_2\otimes g_1"] & X_2 \otimes Y_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow f_2\otimes g_2"] & X_2 \otimes Y_3 \arrow[d] \\
X_3 \otimes Y_1 \arrow[r] & X_3 \otimes Y_2 \arrow[r] & X_3 \otimes Y_3
\end{tikzcd}
=
\begin{tikzcd}
X_1 \otimes Y_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "(f_2 f_1)\otimes(g_2 g_1)"] & X_1 \otimes Y_3 \arrow[d] \\
X_3 \otimes Y_1 \arrow[r] & X_3 \otimes Y_3
\end{tikzcd}
\tag{5.2}
$$

6 Topological Invariants and Quantum Gravity

It is expedient to bear in mind that the representations form a 3-category on one object [19] (the monoidal structure allows us to do this). Thus edges in a state sum get mapped to 2-functors, as seems natural.

The original motivation for this research [3] was to construct a higher categorical version of the Lorentzian Barrett-Crane model [1][20] whose state sum, based on the dual 2-complex, is

$$\mathcal{Z}_{BC} = \sum_{color}^{\int} \prod_{f \in \Delta_2} (\rho_f^2 + n_f^2) \prod_{e \in \Delta_1} \mathcal{A}_e \prod_{v \in \Delta_0} (15j)_v \tag{6.1}$$

where the sum actually contains an integral over the continuous parameter ρ.

We see that the irreps of **Rep**(**Poinc**) contain a choice of orbit in $\mathbf{M}^4$, ie. a radius. This radius can be interpreted as a length, either positive or negative, on an edge in a state sum. The function spaces which appear in the structure of **Rep**(**Poinc**) can be decomposed into representations of $\mathbf{L}$, providing a connection to the Barrett-Crane model. The question of appropriate constraints on the topological theory remains to be studied, but here we outline the basic structure of the new state sum.

The topological Poincaré state sum is constructed by labelling edges with representations from **Meas**, faces with 1-intertwiners and tetrahedra with 2-intertwiners. That is, a functor from a suitable category of triangulated 4-manifolds, or PL pseudomanifolds, to the category **Meas**, a braided version of which will become a 4-category with one object and one 1-morphism.

The amplitude of a 4-simplex is calculated by tracing over the five tetrahedra of its boundary via one of two possible paths

$$+ : \alpha_2 \otimes \alpha_4 \to \alpha_1 \otimes \alpha_3 \otimes \alpha_5 \tag{6.2}$$

$$- : \alpha_1 \otimes \alpha_3 \otimes \alpha_5 \to \alpha_2 \otimes \alpha_4 \tag{6.3}$$

The condition of *sphericity* [21] on **Rep**(**Poinc**) says that these two maps are equal.

Proposition 6.1 **Rep**(**Poinc**) *is a spherical* 2*-category*

We now examine the structure on a tetrahedron in one simple case. First, for edges of a closed triangle

$$\begin{tikzcd} & * \arrow[dr, "E_2"] \arrow[d, phantom, "\Downarrow f"] & \\ * \arrow[ur, "E_1"] \arrow[rr, "E_3"'] & & * \end{tikzcd}$$

labelled by three elementary spacelike irreps, such that (ρ_1, ρ_2, ρ_3) forms an allowable triangle, a 1-intertwiner

$$f : E_1 \otimes E_2 \to E_3$$

is a map $S^2 \to \bullet$. Replacing E_3 with (M_3, S^2) we obtain all equivariant maps $S^2 \to S^2$.

The full labelling of a tetrahedron amounts to the choice of two 1-intertwiners, each a composition of two faces of the tetrahedron, and a choice of 2-intertwiner between them. Tracing over tetrahedra gives 4-simplex labels, which we call $5j$ symbols.

More specifically, in the case of timelike edges we get the set of 2-intertwiners from one copy of the product of two (M_ρ, S^2) to another, corresponding to the two halves of the tetrahedron. This function space decomposes into a tensor product of functions on the orbits in $\mathbf{M}^4$ with the functions on the S^2 fibers. Other choices about the causal structure of the tetrahedron would give analogous results, based on orbit spaces for the stabilizer subgroups $SO(2,1)$, or $E(2)$ in the null case. If we decompose these into harmonics, we will get a combination of relativistic balanced spin nets, as in [1], with ordinary spin nets. The exact form of the connection will contain the constraints for the new state sum, which has yet to be figured out.

The categorical state sum seems to be leading us towards a plausible way of approaching the quantization of the geometry of a triangulated 4-manifold: quantize the edges and configuration spaces of the triangles independently, then constrain them. We would not have hit on this procedure without the 2-category as a guiding framework. It remains to be seen how well it will work.

Thus a 2-categorical topological state sum for the smooth subcategory, formally at least, looks like

$$\mathcal{Z} = \mathcal{N} \sum_{color}^{\int} \prod_{e\in\Delta_1} \rho_e \prod_{f\in\Delta_2} \mathcal{A}_f \prod_{t\in\Delta_3} \mathcal{A}_t \prod_{s\in\Delta_4} (5j)_s \tag{6.4}$$

where $\mathcal{A}_f$ follows from the decomposition of function spaces into representations of $\mathbf{L}$ and their orbit Casimirs, in analogy with the Lorentzian BC model.

7 Conclusions

In the course of working on the structure of **Rep**(**Poinc**), we were surprised to discover how rich it is. A plethora of new possibilities for the construction of state sum models appears to arise. It is natural to speculate on the possible significance of the Lie and crystallographic irreps for features of unification such as phase transitions, or even the insertion of matter.

The application of the new structure to quantum geometry has not been worked out yet, but looks promising. The fibered spaces in the objects we put on parts of triangulations have natural interpretations as staged quantizations of geometrical variables. Problems of regularization remain to be considered, as do the appropriate constraints for quantum gravity.

We have focused on the Poincaré 2-group for reasons of physical interest. In fact, our construction has an analog for any choice of a semisimple Lie group and a representation of it [7]. This means that a whole new chapter of representation theory is opened up, closely allied to a new family of quantum geometries.

The discovery of quantum groups has played a very important role recently in a number of areas of mathematics and physics. Quantum groups can be approached by applying deformation theory to the category of representations of a Lie group. Now we have an interesting 2-category of representations of a 2-group. The techniques of deformation theory have recently been extended to

tensor 2-categories in [22]. It would be interesting to study the deformations of **Rep**(**Poinc**) with, for example, topological invariants in mind.

In [23] it is also observed that for quantum group 1-categories it is really the comodules that should be regarded as the representation theory. The 2-categorical analog is considered, and this would be an interesting avenue for further investigation. The utility of perverse sheaves in understanding quantum groups presumably has a higher dimensional analog, which might be approached from the aforementioned deformation theory.

We believe that when a higher braiding is added, to obtain a 4-category with two singular levels, that it will be possible to begin considering unified models wherein the higher category theory dictates a subtle interplay between massive and spacetime degrees of freedom.

In its modern form, the functorality of Geometric Quantization promises a rigorous approach to the quantum geometry of 4-dimensional state sums. It encompasses, for instance, theorems [24] on the commutativity of quantization and symplectic reduction, and on non-Abelian localization, such as formulae derived by Witten [25] from the path integral of 2-dimensional Yang-Mills theory.

This paper exhibits but one step on the road to an explicit construction of a Poincaré state sum, which is a promising candidate for a physical theory of quantum gravity. It should be pointed out that further categorification is possible in defining 4-dimensional spin foam models. Higher category theory is as yet poorly understood, and in particular there are various definitions of weak n-categories, the relationship between them still somewhat mysterious.

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